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CLOSED FREE HYPERELASTIC CURVES IN THE HYPERBOLIC PLANE AND CHEN–WILLMORE ROTATIONAL HYPERSURFACES*

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ABSTRACT

We show that closed Chen–Willmore rotational hypersurfaces of non-negative curved real space forms are shaped on closed hyperelastic curves of the hyperbolic plane. Then, we study the variational problem associated to this class of curves, proving that there exist a rationally dependent family of closed solutions. They give rise to the first non-trivial examples of Chen–Willmore hypersurfaces in real space forms.

1. Introduction

The total squared mean curvature functional for a surface

$$f\colon \mathbf{M}^2 \to \mathbf{R}^3$$

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in the 3-dimensional Euclidean space is defined by

$$W(f) = \int_M \alpha^2 dA, \tag{1}$$

where α is the mean curvature of the immersion. The study of $W(f)$ was initiated by G. Thomsen in 1923 and W. Blaschke proved in his 1923 book that it is a conformal invariant. Critical points of (1) are usually known as **Willmore surfaces**. Actually, the 1-dimensional version of (1) had been studied much before. In fact, D. Bernoulli introduced the critical curves of $E(\gamma) = \int_\gamma \kappa^2 ds$, κ being the curvature of γ, as a mathematical model for the plane **elastic curves** (or **elasticae**) which were later classified by L. Euler. Much more recently, the problem of the existence and classification of elastic curves in Riemannian manifolds has been a topic of increasing interest (see, for instance, [8], [15], [19]). We shall make use here of the term **hyperelastic** curves (also called n-**elastic** curves), to describe a generalization of the classical elasticae defined as the critical points of $\mathcal{F}^n(\gamma) = \int_\gamma \kappa^n ds$.

T. J. Willmore started in 1968 the study of the so-called **Willmore problem** (the determination of the infimum of $W(f)$ among all immersions of a compact surface of a given topological genus) and proposed the **Willmore conjecture** (the minimum for a topological torus is reached in the conformal class of the Clifford torus and is $2\pi^2$), [10], [25]. These have been topics of intense activity during the last decades. In the early seventies, B-Y. Chen extended the Thomsen–Willmore functional to any submanifold M of any Riemannian manifold. He defined the functional

$$\mathcal{CW}(M) = \int_M (\alpha^2 - \tau_e)^{n/2} dv, \tag{2}$$

α and τ_e being the mean curvature and the extrinsic scalar curvature of M, respectively. It is known as the **Chen–Willmore functional**, it is conformally invariant and its critical points are known as **Chen–Willmore submanifolds**, [9]. In 1978, J. L. Weiner, [24], obtained the Euler–Lagrange equation to be satisfied by **Chen–Willmore surfaces** in a space form of any dimension. In particular, he showed that minimal surfaces of space forms are examples of Chen–Willmore surfaces. Consequently, he used the conformal invariance, the stereographic projection and the Lawson minimal examples in $\mathbf{S}^3$, to produce Willmore surfaces of any genus in $\mathbf{R}^3$. The first examples of Willmore surfaces in $\mathbf{R}^3$, which are not obtained in this way were given by U. Pinkall in 1985, [21], who studied the Willmore tori of $\mathbf{S}^3$ invariant under the Hopf action. J. Langer and D. Singer, [16], characterized the Willmore surfaces of revolution in $\mathbf{R}^3$ in terms of the closed elastic curves of the hyperbolic plane. More generally, D. Ferus and F. Pedit, [12], provided all Willmore tori of $\mathbf{S}^3$ which are invariant under a circle action.

As for the Chen–Willmore tori in $\mathbf{S}^5$, apart from the minimal examples, the first non-trivial example was given by N. Ejiri, [11]. M. Barros and B-Y. Chen, [5], gave a one-parameter family of 2-type Willmore tori in $\mathbf{S}^5$, with constant mean curvature. Also, M. Barros, O. J. Garay and D. Singer, [7], gave examples of Willmore tori in $\mathbf{S}^5$ (both with non-constant mean curvature and with non-zero constant mean curvature) by lifting via the Hopf map closed elastic curves in the real and complex projective planes, respectively. More examples of Willmore surfaces in $\mathbf{S}^5$ and $\mathbf{S}^7$ have been obtained in [18].

The first examples of Chen–Willmore submanifolds of dimension greater than 2 were given by M. Barros and O. J. Garay in [6], where they obtained examples of Willmore 4-dimensional submanifolds in $\mathbf{S}^7$ with non-zero constant mean curvature by lifting closed generalized elastic helices in $\mathbf{S}^3$, via the Hopf map.

Recently, the Euler–Lagrange equation for Chen–Willmore submanifolds of $\mathbf{S}^{n+1}$ has been computed in [17]. (It can be found also in [23] expressed in terms of Moebius invariants.) It seems to be a complicated equation to deal with except in some specific cases, such as minimal surfaces, hypersurfaces of constant mean curvature, and isoparametric hypersurfaces. Using it, the authors show in [13] that, in contrast with the situation for surfaces, not every minimal submanifold of the sphere is a Chen–Willmore submanifold. They also determine the Riemannian products of standard spheres which are Chen–Willmore hypersurfaces of $\mathbf{S}^{n+1}$ (standard examples).

In this paper, we produce the first examples of Chen–Willmore hypersurfaces of $\mathbf{R}^{n+1}$ and $\mathbf{S}^{n+1}$, which are not in the conformal class of the standard examples. We use the conformal invariance of the Chen–Willmore functional and the Palais symmetric criticality principle, to characterize in §2 the Chen–Willmore rotational hypersurfaces of $\mathbf{R}^{n+1}$ and $\mathbf{S}^{n+1}$ in terms of the closed hyperelastic curves (or n-elastic curves) in the hyperbolic plane $\mathbf{H}^2(-1)$. By means of this reduction procedure, the computation of the Euler–Lagrange equations for the Chen–Willmore problem is simplified to the computation of the corresponding Euler–Lagrange equation for hyperelastic curves in $\mathbf{H}^2(-1)$. This computation is carried out in §3, not only for the hyperbolic plane but also for hyperelastic curves lying in a 2-dimensional real space form. The Euler–Lagrange equation is expressed in terms of the curvature of the critical point, and then we prove that there are no periodic solutions, other than geodesics, either in the 2-sphere or in the euclidean plane, but that we have indeed periodic solutions in the hyperbolic plane. In order to obtain examples of compact Chen–Willmore rotational hypersurfaces, we must prove the existence of closed hyperelastic curves in $\mathbf{H}^2(-1)$, which is not guaranteed a priori by the periodic curvature. We find a closure condition in terms of the invariants of the critical curves and show that it is fulfilled by a rationally dependent family of curves which provides the required examples. For a fixed dimension n there is a unique constant curvature closed n-elastic curve, ϵ_n, in $\mathbf{H}^2(-1)$. It is proved also that among the multiple covers of ϵ_n, only the one and two covers are stable with respect to $\mathcal{F}^n$. We also have a qualitative description of the non-constant curvature closed n-elastic curves: they are convex curves traveling along ϵ_n which oscillate between two concentric circles and close up after an integer number of trips around ϵ_n. Getting concrete examples would require first to solve explicitly the Euler–Lagrange equations and then to quantify the closure condition. Although this task does not seem to be possible in general, it has been done for $n = 2$, [16], and for $n = 3$, [4].

2. Chen–Willmore rotational hypersurfaces

A quite general procedure to construct Chen–Willmore submanifolds in warped product Riemannian manifolds has been described in [2]. The next construction can be made following the lines of such a general situation, so we omit the details. In this paper $\mathbf{M}^{n+1}(c)$ will be used to denote either the Euclidean $(n+1)$-space for $c = 0$, $\mathbf{M}^{n+1}(0) = \mathbf{R}^{n+1}$, or the round unit $(n+1)$-sphere for $c = 1$, $\mathbf{M}^{n+1}(1) = \mathbf{S}^{n+1}(1)$. We remove a certain geodesic $\mathbf{L}$ of $\mathbf{M}^{n+1}(c)$. By using suitable coordinates we can identify $\mathbf{M}^{n+1}(c) - \mathbf{L}$, with the product $\mathbf{H}^2(c) \times \mathbf{S}^{n-1}(1)$, where

$$\mathbf{H}^2(0) = \{(u,v) \in \mathbf{R}^2 \mid v > 0\}, \quad \mathbf{H}^2(1) = \{(u,v,w) \in \mathbf{S}^2(1) \mid v > 0\}$$

and $\mathbf{S}^{n-1}(1)$ stands for the unit $(n-1)$-sphere. Let g_o be the standard metric on the half-space $\mathbf{H}^2(c)$ and denote by $d\sigma^{n-1}$ the radius one round metric on $\mathbf{S}^{n-1}(1)$. Then, the standard metric, $\bar{g}_o$, on $\mathbf{M}^{n+1}(c) - \mathbf{L}$ can be written as

$$\bar{g}_o = g_o + f(v)^2 d\sigma^{n-1},$$

where $f\colon \mathbf{H}^2(c) \to \mathbb{R}$ is the projection on the v-axis. In other words, the space $\mathbf{M}^{n+1}(c) - \mathbf{L}$ is nothing but the warped product $\mathbf{H}^2(c) \times_{f(v)} \mathbf{S}^{n-1}(1)$, where $f(v)$ is a positive smooth function defined on the half-space $\mathbf{H}^2(c)$, playing the role of warping function.

For any immersed curve $\gamma\colon [0, L] \to \mathbf{H}^2(c)$, we have the hypersurface $\mathcal{T}_\gamma = \gamma \times_{f(v)} \mathbf{S}^{n-1}(1)$ of $\mathbf{M}^{n+1}(c)$, and we will refer to $\mathcal{T}_\gamma$ as the **rotational**

hypersurface shaped on γ. Let $\mathcal{G} = SO(n)$ be the group of isometries of $(\mathbf{S}^{n-1}(1), d\sigma^{n-1})$. It is clear that the rotational hypersurfaces $\mathcal{T}_\gamma$ are precisely the $\mathcal{G}$-invariant hypersurfaces of $(\mathbf{M}^{n+1}(c) - \mathbf{L}, \bar{g}_o)$ under the obvious action of $\mathcal{G}$. Let $\mathcal{H}$ be the smooth manifold of compact hypersurfaces of $\mathbf{M}^{n+1}(c) - \mathbf{L}$. Then $\mathcal{G}$ also defines a natural action on $\mathcal{H}$ for which the subset of symmetric points is given by

$$\mathcal{H}_\mathcal{G} = \{\mathcal{T}_\gamma \mid \gamma \text{ is a curve immersed in } \mathbf{H}^2(c)\}.$$

The Chen–Willmore functional, $\mathcal{CW}\colon \mathcal{H} \to \mathbb{R}$, is defined as [9]

$$\mathcal{CW}(M) = \int_M (\alpha^2 - \tau_e)^{n/2} dv, \tag{3}$$

where α and τ_e denote the mean curvature and the extrinsic scalar curvature functions of the hypersurface M, and dv is the volume element associated with the induced metric on M. The earlier functional (2) is invariant under the above $\mathcal{G}$-action. Therefore, we can apply the symmetric criticality principle here [20], to characterize those critical points of $\mathcal{H}$ that are $\mathcal{G}$-invariant as the critical points of the restriction of the functional to $\mathcal{H}_\mathcal{G}$. Moreover, it is known that the Lagrangian and the corresponding variational problem are invariant under conformal changes in the background metric, [9], [10]. Taking advantage of this, we compute $\mathcal{CW}$ on $\mathcal{H}_\mathcal{G}$ by making the following conformal change in the metric of $(\mathbf{M}^{n+1}(c) - \mathbf{L}, \bar{g}_o)$:

$$\bar{h}_0 = \frac{1}{f(v)^2}\bar{g}_0 = \frac{1}{f(v)^2} g_o + d\sigma^{n-1}.$$

Now, we observe that $(\mathbf{H}^2(c), \frac{1}{f(v)^2} g_o)$ is nothing but the hyperbolic plane with constant curvature -1. Therefore, $(\mathbf{M}^{n+1}(c) - \mathbf{L}, \bar{h}_0)$ is the Riemannian product of a hyperbolic plane with a round unit $(n-1)$-sphere. Using this, one can prove that the extrinsic scalar curvature, τ_e, of the conformal image of any rotational hypersurface $\mathcal{T}_\gamma$ vanishes identically, and that its mean curvature function, α, is related to the curvature function, κ, of the curve γ in the hyperbolic plane, as follows:

$$\alpha^2 = \frac{1}{n^2}\kappa^2.$$

Hence the restriction of $\mathcal{CW}$ to the space of symmetric points is given by

$$\mathcal{CW}(\mathcal{T}_\gamma) = \frac{\varpi_{n-1}}{n^n} \int_\gamma \kappa^n ds, \tag{4}$$

ϖ_{n-1} being the volume of $\mathbf{S}^{n-1}(1)$.

On the other hand, we may consider the following **curvature energy** action defined on a suitable space of curves of a Riemannian manifold:

$$\mathcal{F}^n(\gamma) = \int_\gamma \kappa^n ds, \tag{5}$$

where κ denotes the curvature function of γ. Observe that for $n = 2$ this is nothing but the classical Bernoulli's elastic functional, [15]. For this reason, we refer to the critical points of (5) as **free hyperelastic curves**, or more accurately as **free n-elastic curves.**

Therefore, by using this notation, we obtain the following reduction of variables result as a consequence of (4):

2.1 PROPOSITION: *A rotational hypersurface T_γ is a Chen–Willmore hypersurface in $\mathbf{M}^{n+1}(c)$, $c = 0, 1$, if and only if γ is a free n-elastic curve in the hyperbolic plane.*

Hence in order to produce examples of closed Chen–Willmore rotational hypersurfaces, we must find closed n-elastic curves in the hyperbolic plane. If $n = 2$ (classical elasticae), the Euler–Lagrange equations of (5) can be explicitly integrated, its closed critical points can be classified and their stability analyzed, [15]. This provides us with a deep control on the Willmore surfaces of revolution in $\mathbf{R}^3$, which has been used by Langer and Singer to confirm the Willmore conjecture for surfaces of revolution in $\mathbf{R}^3$, [16]. For $n = 3$, the Euler–Lagrange equations of (5) can be also explicitly integrated and its closed critical points can be determined, providing nice applications to the study of a natural extension of the Nambu–Goto–Polyakov action and to the determination of the first examples of Chen–Willmore hypersurfaces in $\mathbf{R}^4$, [4]. However, we cannot expect such a good control in the general case since the Euler–Lagrange equations of (5) can rarely be explicitly integrated. The next section is devoted to the analysis of the n-elastic curves variational problem in 2-dimensional real space forms. The analysis of more general curvature dependent energy functionals, defined on curves which lie in the 3-sphere, has been conducted in [1].

3. Closed free n-elasticae in 2-dimensional space forms

Let $\mathbf{M}^2(G)$ denote a 2-dimensional real space form of curvature G, ∇ its Levi-Civita connection and let γ: $\mathbf{I} \longrightarrow \mathbf{M}^2(G)$, $\gamma(t)$, be an immersed C^∞ curve; $V(t)$ will denote the tangent vector to $\gamma(t)$, $T(t)$, $N(t)$ unit tangent and normal vectors, respectively, and $v = \langle V(t), V(t)\rangle^{1/2}$ its speed. We will write $\kappa(t)$ for the oriented curvature of $\gamma(t)$ and Γ: $(-\epsilon, \epsilon) \times \mathbf{I} \longrightarrow \mathbf{M}^2(G)$, $\Gamma(w, t)$, for a variation with $\Gamma(0,t)=\gamma(t)$ and variation vector field $W(t)=\frac{\partial\Gamma}{\partial w}(0,t)$ along the curve $\gamma(t)$. More generally we use $T(w,t)$, $W(w,t)$, $V(w,t)$ etc., with the usual meaning. For a fixed natural number r we consider the functional

$$\mathcal{F}^r(\gamma)=\int_\gamma \kappa^r ds=\int_0^1 \kappa^r v dt, \tag{6}$$

where s denotes the arclength of γ. As we said before, critical points of $\mathcal{F}^r$ will be called free r-hyperelastic curves (or free r-elastic curves). We use the letter r to emphasize that it is not necessarily a natural number.

Our approach to the study of (6) is inspired by [15] where they take care of the case $r=2$. However, since we cannot explicitly integrate the Euler–Lagrange equation for a generic r, we must use a different argument to find periodic solutions. In contrast with the situation for $r=2$, there are no closed r-elastic curves in the 2-dimensional sphere if $r>2$.

We define the following vector fields along γ:

$$\begin{aligned}\mathcal{K}&=r\kappa^{r-1}N,\\ \mathcal{J}&=(r-1)\kappa^rT+r(r-1)\kappa^{r-2}\kappa_sN,\\ \mathcal{E}&=\nabla_T\mathcal{J}+R(\mathcal{K},T)T,\end{aligned} \tag{7}$$

where $R(,)$ denotes the Riemannian curvature tensor of $\mathbf{M}^2(G)$.

Then, using standard arguments which involve some integrations by parts, the Frenet equations of γ and a technical lemma similar to lemma 1.1 of [15], one can obtain the first variation formula of this functional,

$$\frac{d\mathcal{F}^r}{dw}(0)=\int_0^L\langle W,\mathcal{E}\rangle ds+\mathcal{B}(W,\gamma)|_0^L, \tag{8}$$

where L is the length of γ and the boundary operator is given by

$$\mathcal{B}(W,\gamma)=\langle\nabla_TW,\mathcal{K}\rangle-\langle W,\mathcal{J}\rangle. \tag{9}$$

The boundary operator can be dropped by assuming appropriate boundary conditions. We are mainly interested in closed critical points, so our space of curves Ω is formed by isometric immersion of $\mathbf{S}^1(1)$ in $\mathbf{M}^2(G)$, that is by the closed regular curves of $\mathbf{M}^2(G)$. Thus under suitable boundary conditions, γ is a critical point of $\mathcal{F}^r$ if and only if the following Euler–Lagrange equation is satisfied:

$$\mathcal{E}(\gamma)=r(r-1)\kappa^{r-3}(\kappa\kappa_{ss}+(r-2)\kappa_s^2+\frac{\kappa^4}{r}+G\frac{\kappa^2}{r-1})=0. \tag{10}$$

If $r = 1$, then every curve in the Euclidean plane is a critical point and there are no critical points either in the 2-dimensional sphere or in the hyperbolic plane. This case has been more generally considered in [3]. If $r = 2$, then we have the classical Euler–Bernoulli elasticae. This case has been extensively studied in the literature as we have pointed out. Assume $r > 2$. Then geodesics are trivial solutions of equation (10). If κ is a non-zero constant, then $\kappa^2 = -rG/(r-1)$, hence we only have non-zero critical circles when $G < 0$.

Assume that κ is a non-constant. Now we want to obtain a first integral of (10). A vector field W is called a **Killing field** along γ, if for any variation in the direction of W we have $\partial v/\partial w = \partial \kappa/\partial w = 0$, where v is the speed of γ. It was proved in [15] that a Killing field along a curve γ is the restriction to γ of a Killing field on $\mathbf{M}^2(G)$. First, we observe that (10) is equivalent to

$$\nabla_T \mathcal{J} = -rG\kappa^{r-1}N, \tag{11}$$

and then

$$\langle \nabla_T \mathcal{J}, T\rangle = \langle \nabla_T^2 \mathcal{J} + G\mathcal{J}, N\rangle = 0, \tag{12}$$

which, using lemma 1.1 of [15], can be seen to be equivalent to $\partial v/\partial w = \partial \kappa/\partial w = 0$. Thus, we have

3.1 Proposition: *Let γ be a free r-hyperelastic curve in $\mathbf{M}^2(G)$ and $\{T, N\}$ its Frenet frame. Then the vector field $\mathcal{J} = (r-1)\kappa^r T + r(r-1)\kappa^{r-2}\kappa_s N$ is a Killing vector field along γ. Therefore, $\mathcal{J}$ is the restriction to γ of a Killing field on $\mathbf{M}^2(G)$ (which we also denote by $\mathcal{J}$).*

Now one can apply Noether's argument relating symmetries of $\mathcal{F}^r$ to constancy of motion along γ. If γ is a critical point for $\mathcal{F}^r$ under any boundary conditions, then $\mathcal{E}(\gamma) = 0$. Thus the first variation formula can be written as

$$\frac{d\mathcal{F}^r}{dw}(0) = \mathcal{B}(W,\gamma)|_0^L = \langle \nabla_T W, \mathcal{K}\rangle - \langle W, \mathcal{J}\rangle|_0^L. \tag{13}$$

Therefore, taking $W = \mathcal{J}$ in the above formula and noticing that the variation formula remains valid for any $0 < t < L$, we get

3.2 Proposition: *Let γ be a free r-hyperelastic curve in $\mathbf{M}^2(G)$. Let us denote by $\{T, N\}$ its Frenet frame and let $\mathcal{J}$, $\mathcal{K}$ be given as in (7). Then $\langle \nabla_T \mathcal{J}, \mathcal{K}\rangle - \langle \mathcal{J}, \mathcal{J}\rangle$ is constant along γ.*

As a consequence of the above Proposition and (7), we have that the curvature of a critical free closed r-elastic curve must satisfy

$$r^2(r-1)^2\kappa^{2r-4}\kappa_s^2 + (r-1)^2\kappa^{2r} + r^2G\kappa^{2r-2} = d. \tag{14}$$

If $G \geq 0$, then $d > 0$ and κ has no zeros and we may assume it is positive. If κ were a smooth periodic solution of (14), then we would have that κ varies between its minimum $\kappa(s_1) = \alpha_1 > 0$ and its maximum $\kappa(s_2) = \alpha_2 > 0$. From (14) we see that α_1, α_2 should be two different positive solutions of the polynomial $q_d(x) = d - (r-1)^2 x^{2r} - r^2 G x^{2r-2}$. But $q_d(x)$ has only one positive solution of $d > 0$, thus (14) has no periodic solutions for $r > 2$. In particular, we have

3.3 Proposition: *There are no closed free r-elasticae in $\mathbf{R}^2$, $r > 2$. Moreover, great circles are the only closed free r-elasticae in the 2-dimensional sphere $\mathbf{S}^2(G)$ for $r > 2$.*

Observe that any closed curve in $\mathbf{R}^2$ is a free 1-elastica and that there are no free 1-elasticae in $\mathbf{S}^2(G)$ as we mentioned before, [3]. Also, there are no closed free 2-elasticae (classical elasticae) in $\mathbf{R}^2$ and there are plenty of them in $\mathbf{S}^2(G)$, [15].

It remains to study the case of the hyperbolic plane. It suffices to consider $G = -1$, that is $\mathbf{M}^2(G) = \mathbf{H}^2(-1)$. Assume that $\kappa > 0$; then we write the Euler–Lagrange equation (10) as an autonomous system

$$\begin{aligned} \dot{z} &= f(z), \\ z &= (\kappa, \kappa_s), \\ f\colon D &= R^+ \times R \longrightarrow R^2. \end{aligned} \tag{15}$$

We see that $a = ((\frac{r}{r-1})^{1/2}, 0)$ is a critical point of (15) and that the equation of the orbits in the phase space can be obtained from (14),

$$r^2(r-1)^2 x^{2r-4} y^2 + (r-1)^2 x^{2r} - r^2 x^{2r-2} = d. \tag{16}$$

Thus periodic solutions of (10) would correspond to closed orbits in D. The function

$$F\colon \mathbf{R}^2 \longrightarrow \mathbb{R}$$

$$F(x,y) = r^2(r-1)^2 x^{2r-4} y^2 + (r-1)^2 x^{2r} - r^2 x^{2r-2}$$

is a first integral of equation (15). We also have

$$F_{xx}(a) = \frac{2r^{2r-2}(4r-3)}{(r-1)^{2r-4}}, \quad F_{yy}(a) = \frac{2r^{2r-2}}{(r-1)^{2r-6}} \quad \text{and} \quad F_{xy}(a) = 0;$$

then a is a non-degenerate critical point of F of index zero. Hence by using Morse's Lemma, we can find closed orbits in a neigborhood of a and, therefore, we have periodic solutions of (10). Actually, if we denote $q(x) = (r-1)^2 x^{2r} - r^2 x^{2r-2}$, then we see that the only positive root of $q(x)$ is $r/(r-1)$ and that it reaches its minimum value $-r^r/(r-1)^{r-1}$ at $x = r^{1/2}/(r-1)^{1/2}$. Hence for any $d \in \mathbb{R}$ such that $-r^r/(r-1)^{r-1} < d < 0$, the orbits given by (16) are closed and the corresponding solutions of (10), $\kappa_d(s)$, are periodic. Denoting $q_d(x) = d - q(x)$, one has that the polynomial $q_d(x)$ has two simple positive roots $\alpha_1^d < \alpha_2^d$, and the corresponding periodic solution $\kappa_d(s)$ varies between its minimum $\kappa_d(s_1) = \alpha_1^d$ and its maximum $\kappa_d(s_2) = \alpha_2^d$. Also, $\kappa_d(s)$ is symmetric with respect to $s = s_2$.

Take one of these periodic solutions $\kappa_d(s)$ and denote by $\gamma_d(s)$ the curve with curvature $\kappa_d(s)$ in $\mathbf{H}^2(-1)$ (unique up to isometries). We need to establish conditions under which $\gamma_d(s)$ is a closed curve. In order to do so, we make use again of the Killing field $\mathcal{J}$. We know that $\mathcal{J}$ is a Killing field along the critical point $\gamma_d(s)$. Assume that at $p_o = \gamma_d(s_o)$ we have a vertex, $\kappa_d(s_o) = 0$. Then $\mathcal{J}$ is tangent to $\gamma_d(s)$ at p_o and it can be extended to a Killing field on the whole hyperbolic plane, which we also denote by $\mathcal{J}$. Let β be the integral curve of $\mathcal{J}$ passing by p_o at time t_o. Denote by $\overline{T}$ the unit tangent vector to β. From (7) we have

$$\frac{d\|\mathcal{J}\|^2}{ds} = 2r^2(r-1)\kappa_s\kappa^{2r-3},$$

which means that $d\|\mathcal{J}\|/ds = d(\|\mathcal{J}\|)^{-1}/ds = 0$ at p_o. Using this and the Euler–Lagrange equation (10) we get

$$\begin{aligned}(\nabla_{\overline{T}}\overline{T})(p_o) = \Big(\frac{D}{dt}\frac{\mathcal{J}}{\|\mathcal{J}\|}\Big)(t_o) &= \Big(\frac{D}{ds}\frac{\mathcal{J}}{\|\mathcal{J}\|}\Big)(s_o)\\ &= \frac{1}{\|\mathcal{J}\|}\Big(\frac{D}{ds}\mathcal{J}\Big)(s_o)\\ &= \frac{r}{(r-1)\kappa(s_o)}N(s_o)\end{aligned}$$

and therefore the curvature of β is $\kappa_\beta = r/(r-1)\kappa(s_o)$. On the other hand, $\kappa_d(s)$ varies between $\kappa_d(s_1) = \alpha_1^d$ and $\kappa_d(s_2) = \alpha_2^d$. When $-r^r/(r-1)^{r-1} < d < 0$ we have $0 < \alpha_2^d < r/(r-1)$. Thus $\kappa_\beta > 1$ and β is a circle. Hence

3.4 Proposition: *Let γ be a convex free r-hyperelastic curve in $\mathbf{H}^2(-1)$. Then the extension of $\mathcal{J}$ to the hyperbolic plane is a rotational Killing field.*

Choose the center of $\mathcal{J}$ as the origin of the Poincaré disc model of $\mathbf{H}^2(-1)$ and take polar coordinates $x(\theta,\varphi)$ so that $b\mathcal{J} = x_\theta$. By using (7) and (14), we have

$$\theta_s = \frac{\langle T, x_\theta\rangle}{\|x_\theta\|^2} = \frac{(r-1)\kappa^r}{b(d + r^2\kappa^{2r-2})}. \tag{17}$$

Moreover, one can choose b so that $b^2 d = -1$. Hence, one gets from (17)

3.5 PROPOSITION: *Let γ_d be a convex free r-hyperelastic curve in $\mathbf{H}^2(-1)$ whose curvature is given by a periodic solution of (10), κ_d, with period h_d. Then γ_d is a closed curve if and only if its rotation in one period of the curvature Λ_d is a rational multiple of π. In other words, γ_d closes up if and only if*

$$\Lambda_d := \sqrt{-d}\int_0^{h_d} \frac{(r-1)\kappa^r ds}{(d+r^2\kappa^{2r-2})} = q\pi \tag{18}$$

for some rational number q.

Now, consider a number d such that $-r^r/(r-1)^{r-1} < d < 0$ and γ_d is as in the above Proposition. Its curvature, $\kappa_d(s)$, varies between $\kappa_d(s_1) = \alpha_1^d$ and $\kappa_d(s_2) = \alpha_2^d$ and is symmetric with respect to $s = s_2$. Also, $d\kappa_d/ds \neq 0$ in (s_1, s_2). Thus using (14) one gets

$$\begin{aligned}\Lambda_d =&\sqrt{-d}\int_0^{h_d} \frac{(r-1)\kappa^r ds}{(d+r^2\kappa^{2r-2})} \\ =&2\sqrt{-d}\int_{\alpha_1^d}^{\alpha_2^d} \frac{r(r-1)^2\kappa^{2r-2}}{(d+r^2\kappa^{2r-2})}\frac{d\kappa}{\sqrt{q_d(\kappa)}},\end{aligned} \tag{19}$$

where $q_d(x) = d-(r-1)^2x^{2r}+r^2x^{2r-2}$. If d belongs to $(-r^r/(r-1)^{r-1}, 0)$, one sees that $d+r^2\kappa^{2r-2}\neq 0$ and $q_d(x)$ has two simple positive roots $\alpha_1^d < \alpha_2^d$. Hence, from (19) we have Λ_d is finite and moves continuously in an interval as d moves in $(-r^r/(r-1)^{r-1}, 0)$. Therefore, corresponding to the solutions of (18), we have

3.6 PROPOSITION: *For any natural number $r > 2$ there exists an infinite family of closed free r-hyperelastic curves in $\mathbf{H}^2(-1)$.*

A description of the closed solutions can be given as follows. Choose a real number $d \in (-r^r/(r-1)^{r-1}, 0)$ such that $\Lambda_d = 2\pi\frac{m}{n}$, take κ_d the corresponding periodic solution of (10) and let γ_d be the associated closed r-elastic curve in $\mathbf{H}^2(-1)$, as guaranteed by Proposition 3.6. Let $0 < \alpha_1^d < \alpha_2^d$ be the minimum and maximum values of κ_d. Let us denote by ϵ_r, ζ_r^1, ζ_r^2 the circles of $\mathbf{H}^2(-1)$ with curvatures $r^{1/2}/(r-1)^{1/2}$, $r/(r-1)\sqrt{\alpha_1^d}$, $r/(r-1)\sqrt{\alpha_2^d}$, respectively. Then γ_d is a convex curve which oscillates between ζ_r^1, ζ_r^2 (touching them tangentially) and closes up after n periods of its curvature and m trips around ϵ_r.

Combining Proposition 3.6 and Proposition 2.1, one gets

3.7 PROPOSITION: *For any natural number $n \in \mathbb{N}$, $n > 1$, there exists an infinite family of closed Chen–Willmore hypersurfaces in $\mathbf{M}^{n+1}(\mathbf{c})$. They are the rotational hypersurfaces shaped on the closed free n-hyperelastic curves in $\mathbf{H}^2(-1)$ given in the above proposition.*

A remaining interesting question concerns the stability of the critical points. Here, we consider the only circle ϵ_r, which is a critical point of $\mathcal{F}^r$ in $\mathbf{H}^2(-1)$, and denote by δ its curvature, where $\delta^2 = r/(r-1)$. Now, let us denote by ϵ_r^m the m-fold circle of curvature δ. As for the stability, we have

3.8 PROPOSITION: *ϵ_r^m is stable for every $m = 1, 2$ and unstable otherwise.*

Proof: We consider a closed critical point γ of $\mathcal{F}^r$. Denote by κ its curvature and take a normal variation of γ with variation normal field $W = \phi N$. Then, computing the second variation formula at γ and using (8), we obtain

$$\begin{aligned} \frac{d^2\mathcal{F}^r}{dt^2}(0) &= \int_\gamma \langle W, \nabla_W \mathcal{E}\rangle ds \\ &= \int_\gamma \mu_1 W(\kappa) + \mu_2 W(\kappa_s) + \mu_3 W(\kappa_{ss}), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \mu_1 =& r(r-1)(r-2)\kappa^{r-4}(\kappa\kappa_{ss} + (r-3)\kappa_s^2)\phi \\ &+ \kappa^2 r(r-1)\kappa^{r-2}(\kappa^2(1+1/r)-1)\phi, \\ \mu_2 =& 2r(r-1)(r-2)\kappa_s\kappa^{r-3}\phi, \\ \mu_3 =& r(r-1)\kappa^{r-2}\phi, \end{aligned}$$

and

$$\begin{aligned} W(\kappa_{ss}) =& \phi_{ssss} + \phi_{ss}(\kappa^2-1) + 5\kappa\kappa_s\phi_s + (3\kappa_s^2 + 4\kappa\kappa_{ss})\phi, \\ W(\kappa_s) =& \phi_{sss} + \phi_s(\kappa^2-1) + 3\phi\kappa\kappa_s, \\ W(\kappa) =& \phi_{ss} + \phi(\kappa^2-1). \end{aligned}$$

Now, evaluating at $\gamma = \epsilon_r^m$ and integrating by parts,

$$\frac{d^2\mathcal{F}^r}{dt^2}(0) = \left(\frac{r}{r-1}\right)^{r/2} \int_{\epsilon_r^m} ((r-1)\phi_{ss} + 2\phi)((r-1)\phi_{ss} + \phi) ds. \tag{21}$$

Puting $p = \sqrt{r-1}$, we see that ϕ must have a Fourier series of the form

$$\frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{ns}{mp}\right) + b_n \sin\left(\frac{ns}{mp}\right).$$

Therefore, from the above formula one obtains

$$\frac{d^2\mathcal{F}^r}{dt^2}(0) = \frac{\pi m r^{r/2}}{(r-1)^{(r-1)/2}} \left\{ a_o^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\left(\frac{n^2}{m^2} - 1\right)\left(\frac{n^2}{m^2} - 2\right) \right\}. \tag{22}$$

If $m > 2$, then there are choices of $W = \phi N$ which makes (22) negative and then ϵ_r^m is unstable. On the other hand, if $m \leq 2$ then the second variation is positive semidefinite. Equality occurs when $a_n = b_n = 0$, $n \neq m$ and $\phi = a_m \cos(s/r) + b_m \sin(s/r)$, which corresponds to the space of Killing fields along γ. ∎

We don't know whether or not they are the only stable critical points of $\mathcal{F}^r$ as proved for $r = 2$ in [15].

4. Numerical experiments

For the sake of simplicity, in this section we omit details concerning the numerics involved in the following computations and graphic representations.

4.1 EXPERIMENT 1: *Uniqueness of the closed solutions.*

For any given $r \in \mathbb{N}$, the space of periodic solutions of (14) is parametrized in $(-r^r/(r-1)^{r-1}, 0)$. Since both the non-zero extreme of this interval and the angular progression defined in (18) depend on r, we denote them in this section by $d(r)$ and Λ_d^r, respectively.

Experiments suggest that Λ_d^r is a monotonically decreasing function which varies from $\sqrt{2}\pi$ to π on $(-r^r/(r-1)^{r-1}, 0)$ for any $r \geq 2$. This is known to be true for $r = 2$ and 3, [15], [4]. Formal confirmation of this fact would result in a more accurate statement of Proposition 3.6: For any $r > 1$, and for any couple of natural numbers m, n satisfying $m > 1$, $\frac{1}{2} < m/n < 1/\sqrt{2}$, there exist a *unique* critical point of $\mathcal{F}^r$ in $\mathbf{H}^2(-1)$ which closes up after n periods of its curvature and m trips around ϵ_r.

We have computed numerically these functions for several values of r and have collected graphically part of this information in Figure 1.

4.2 EXPERIMENT 2: *A confirmation of the generalized Willmore conjecture for rotational hypersurfaces.*

Let us consider the standard embeddings of the Riemannian product of two spheres as a hypersurface in the unit sphere

$$x\colon \mathbf{S}^m\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbf{S}^{n-m}\left(\sqrt{\frac{m}{n}}\right) \to \mathbf{S}^{n+1}(1)$$

(here, the number between brackets is the radius of the corresponding sphere). They are known as the **standard examples** W_m^n and they are stable Chen–Willmore hypersurfaces, [24], [13].

The following generalization of the Willmore conjecture was proposed in [13], [22], [14]: For any $1 \leq m \leq n-1$, let G_m be an n-dimensional manifold diffeomorphic to $\mathbf{S}^m \times \mathbf{S}^{n-m}$ and let $x\colon G_m \to \mathbf{S}^{n+1}(1)$ an isometric immersion.

Then

$$\mathcal{CW}(G_m) = \int_{G_m} (\alpha^2 - \tau_e)^{n/2} dv \geq \frac{(n-m)^{m/2} m^{(n-m)/2}}{n^{n/2}(n-1)^{n/2}} \varpi_{n-m}\varpi_m, \tag{23}$$

ϖ_m being the volume of $\mathbf{S}^m(1)$. Equality holds if and only if G_m is Moebius equivalent to the standard examples W^n_m. Then, for a hypersurface with the topological type of rotational hypersurfaces, $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, formula (23) turns out to be

$$\mathcal{CW}(G_1) \geq \frac{2\pi\varpi_{n-1}}{n^{n/2}(n-1)^{(n-1)/2}}. \tag{24}$$

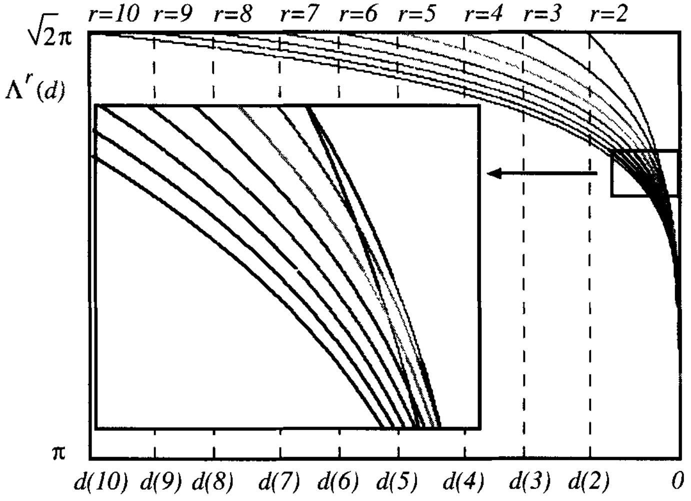

Figure 1. Monotonicity of Λ^r.

Assume first that ϱ_t is a circle of radius t in $\mathbf{H}^2(-1)$. Then

$$\mathcal{F}^n(t) = \int_{\varrho_t} \kappa^n ds = 2\pi \frac{\cosh^n(t)}{\sinh^{n-1}(t)},$$

which reaches its minimum value $\eta_n = 2\pi\sqrt{n^n/(n-1)^{n-1}}$ at ϵ_n (the circle of curvature $\sqrt{n/(n-1)}$). On the other hand, we consider the function $\widehat{\mathcal{F}}^n(d)\colon = \int_o^{h_d} \kappa_d^n ds$, where h_d is the period of the curvature κ_d, which corresponds to the critical point γ_d of $\mathcal{F}^n$ associated to a certain $d \in (-n^n/(n-1)^{n-1}, 0)$. Numerical computations show that for a given $n \in \mathbb{N}$, $\widehat{\mathcal{F}}^n(d)$ is a monotonically decreasing function of d whose lowest upper bound is given by

$$\lim_{d \to -n^n/(n-1)^{n-1}} \widehat{\mathcal{F}}^n(d) = \frac{1}{\sqrt{2}}\eta_n.$$

Since γ_d needs at least two periods in order to close up, this would mean that the minimum value of $\mathcal{F}^n$ on critical points is reached at ϵ_n, suggesting that it is the absolute minimum of $\mathcal{F}^n$ on closed curves of $\mathbf{H}^2(-1)$. Then, one may use (4) to obtain a confirmation of the generalized Willmore conjecture for rotational-type hypersurfaces. The above arguments are known to be true for $n = 2$, [16].

Figure 2 shows graphical manipulation of the data for different values of n.

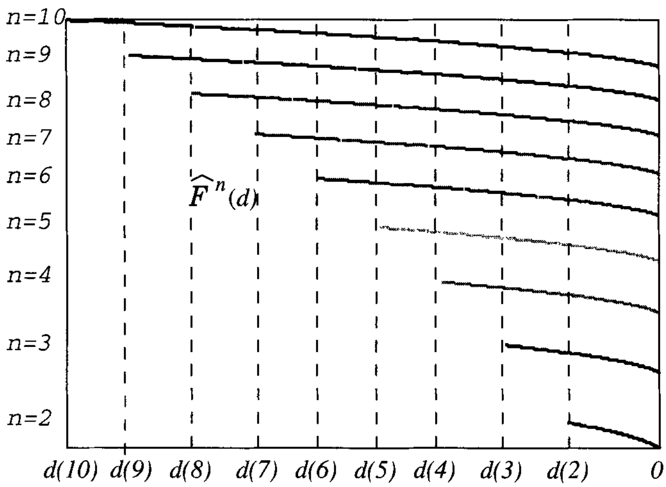

Figure 2. Willmore energies.

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